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Constructing analytic approximate solutions to the Lane–Emden equation

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ABSTRACT

We derive analytic approximations to the solutions of the Lane–Emden equation, a basic equation in Astrophysics that describes the Newtonian equilibrium structure of a self-gravitating polytropic fluid sphere. After recalling some basic results, we focus on the construction of rational approximations, discussing the limitations of previous attempts, and providing new accurate approximate solutions.

1. Introduction

Polytropic fluid sphere models are ubiquitous in Astrophysics. They have been instrumental in the development of stellar structure theory [1], as well as in the investigation of the dynamics of spherical galaxies and star clusters [2]. Gaining insight into their equilibrium and stability properties is therefore an important task that has attracted, and still attracts much interest.

Polytropic models are characterized by a simple equation of state, $p \propto \rho^{(n+1)/n}$, with p and ρ the fluid pressure and density, respectively, and n the so-called polytropic index. In an isotropic configuration, the Newtonian, hydrostatic equilibrium structure of the fluid sphere is then determined by a second order, generally nonlinear ordinary differential equation for the gravitational potential,

$$y'' + \frac{2}{x} y' = -y^n \,, \tag{1}$$

known as the Lane–Emden equation [LEE hereafter; here $y^n = \rho(x)/\rho(0)$, x is a scaled radial coordinate, and the prime denotes derivation with respect to x]. The problem is completed by the boundary conditions

$$y(0) = 1, \qquad y'(0) = 0 \,, \tag{2}$$

which ensure regularity at the sphere center. For $0 \le n < 5$, the solutions of the boundary value problem (1)–(2) decrease monotonically with x and vanish at a finite radius x_1 (the star radius, in a stellar context), which is a rapidly increasing function of n ($x_1 \to \infty$ for $n \to 5$).

Exact solutions to the LEE are only known for the linear cases $n = 0, 1$, and for $n = 5$. For other values of n, well-known numerical methods for initial value problems may be used to compute accurate approximate solutions. Reference results have been obtained in [3] and [4], through Runge–Kutta integrations, and in [5], using the Chebyshev pseudospectral method. Analytic approximations have also been sought; classical works are [6], focusing on rational approximations; [7], in which a sophisticated functional ansatz was developed; and [8], where the delta-perturbation method was used to derive an approximation for $x_1(n)$.

In the last decade, the search for approximate solutions to the LEE has produced many papers (see, e.g., the list given in the introduction of [9]), but, apparently, few useful results. A problem is that most of these works restrict to the interval $[0, 1]$, denoted as the "core region" in astrophysical contexts, even though the radial ranges of interest are typically much larger. Consider, for example, the $n = 3$ polytrope, which provides a reasonable description of the Sun's structure, and is consequently widely used as a test case: since its boundary is at $x = x_1 \simeq 6.897$, a useful approximation for the structure of this polytrope should cover a range about *seven* times larger than the core region.

Moreover, in the core region, approximate solutions of any desired accuracy can be easily constructed using conventional Taylor series expansions about the origin, because the convergence range of these series is always significantly larger than unity (see [10]). It is therefore unclear why so many papers in recent years have focused on using more complicated approaches (Adomian decomposition, Homotopy analysis method, Boubaker polynomials, among others; see [9] and references therein) to derive alternative approximations over $[0, 1]$. Often in these papers important works on the properties of series solutions to the LEE, such as [10] and [11], are not cited, and a detailed comparison with relevant previous work is lacking.

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This provided motivation for the present Letter, whose first objective is to recall some basic results that should be taken into account, and used as a reference where appropriate, by anybody seeking new approximate solutions to the LEE. We shall then focus on the construction of rational approximations, clarifying the limitations of previous attempts, and deriving some new, simple and accurate approximate solutions to (1)–(2).

2. Some basic results

a) *Exact solutions.* It is said in [9] that "only the cases $n = 0$, $n = 1$ and $n = 5$ can be solved analytically...". This is probably true, but, as far as we know, it has not yet been proven. It was stated in [12], without a proof, that application of the Lie group analysis shows that (1) is nonintegrable in a closed form for other values of n, because its Lie algebra is zero-dimensional. But then, it was also noted that there are some – albeit rare – cases in which a zero-dimensional Lie algebra does not preclude integrability. Thus, it would seem more prudent to say that the cases $n = 0, 1, 5$ are the only ones that are currently known to be analytically solvable.

b) *Scaling.* In some works, as for example in [13], an apparent generalization of the problem was considered, with the boundary condition on y given by

$$y(0) = a\,, \tag{3}$$

a being a positive constant. It is readily seen, however, that the scaling

$$y = a\tilde{y}, \qquad x = \tilde{x}/a^{(n-1)/2}, \tag{4}$$

maps (1) into an equation of the same form for $\tilde{y}(\tilde{x})$, with $\tilde{y}(0) = 1$, thus reducing the problem to the standard one. Most of the figures of [13] are just illustrations of this scaling; for example, the only difference between the solutions displayed in Figs. 7, 8, and 9 is a scaling factor a for the y-axis and a $1/a$ factor for the x-axis, in agreement with (4).

c) *Series solutions.* Taylor series expansions for y about the origin (up to the x^{10} power) were given in [13], for several values of n. Those series are special cases of the well-known general expansion

$$\begin{aligned} y \simeq 1 &- \frac{1}{3!}x^2 + \frac{n}{5!}x^4 - \frac{n(8n-5)}{3\times 7!}x^6 \\ &+ \frac{n(122n^2-183n+70)}{9\times 9!}x^8 \\ &- \frac{n(5032n^3-12\,642n^2+10\,805n-3150)}{45\times 11!}x^{10} + \ldots \end{aligned} \tag{5}$$

(see, e.g., [4] and references therein). Analytic calculation of higher order terms in (5) is cumbersome, but, when needed, such terms can be easily obtained numerically. We have computed some of them using the stable, coupled recurrence relations for y and ρ given in [10], with the purpose of estimating the accuracy of (5), truncated at the x^{10} term, in the core region. We find that, for $n = 1, 2, 3, 4$, the values of $y(1)$ are correct to $9, 5, 4$, and 3 decimal digits, respectively. Accuracy is of course higher at smaller values of x. Thus, up to $n = 4$, the first six terms in the series expansion (5) yield sufficient accuracy in the core region for most practical purposes.

We note that, since the Taylor series expansion converges in the core region, and can be easily computed with high accuracy, it *should* be used as a benchmark for any alternative approximation over $[0, 1]$.

d) *Convergence of the series solutions.* It has long been known that, for n large enough, the Taylor series expansions about the origin do not cover the whole radial extent of the star (see, e.g., [14] and [15]). More recently, the convergence radius x_s of the series expansion was accurately determined (see [10] and [11]) for several values of n, through non-trivial numerical computations. It was found that x_s is a decreasing function of n, and that the expansion converges over the whole radial extent of the star only for n smaller than about 1.9. For larger values of n, x_s becomes a fraction of x_1: x_s/x_1 is less than $2/5$ for $n = 3$, and only about $2/15$ for $n = 4$. This behavior results from the presence of singularities in the complex plane that were investigated in detail in [11]. Both in [10] and in [11] it was also shown that the singularities may be transformed away through appropriate changes of independent variable. The expansions in the new variables do converge up to the star boundary, albeit quite slowly (very slowly for $n > 3$).

e) *Other approximations.* Despite the long history of the subject, few useful, alternative analytic approximations have been derived that cover the whole radial range. The one constructed in [7] is accurate, and in principle holds for any n, but has a complicated structure, with three coefficients to be fitted, case by case, and an arbitrarily chosen function; optimal coefficients were only computed for $n = 0.5, 1, 1.5, 2, 3$, and for some other n values larger than 5 (see Table 1 of [7]). The $(2, 2)$ Padé approximant computed in [6] was shown to be accurate for $0 \le n \le 2.5$, but its behavior for larger n is unclear. Approximations of a different form were derived in [10], which require a priori knowledge of both x_1 and $y'(x_1)$. The coefficients of these approximations were tabulated for integer and half-integer values of n, in the range $1 \le n \le 4$.

3. Rational approximations

A well-known technique for extending the accuracy of the series expansions beyond their radius of convergence is that of the Padé approximants, which are rational approximations constructed from the Taylor series (see, e.g., [16]). In the context of the LEE, this approach was first pursued in [6], where the first two diagonal Padé approximants, $y_n^{(1,1)}$ and $y_n^{(2,2)}$, were computed. The second Padé approximant, which results from imposing the first four terms in the Taylor series expansion around the origin, was written as

$$y_n^{(2,2)} = \frac{a_1 + a_2x^2 + a_3x^4}{b_1 + b_2x^2 + b_3x^4}\,, \tag{6}$$

with

$$\begin{aligned} a_1 &= b_1 = 45\,360(17n-50)\,, \\ a_2 &= 420(178n^2-951n+1250)\,, \\ a_3 &= 1290n^3-10\,849n^2+29\,100n-24\,500 \\ b_2 &= 420(178n^2-645n+350)\,, \\ b_3 &= 15n(86n^2-321n+190)\,. \end{aligned} \tag{7}$$

For $0 \le n \le 2.5$, it provides a good approximation over the whole radial extent of the star, because it yields fairly accurate values of the star radius, with a maximum error, at $n = 2.5$, of about 1.7%. The behavior for larger n was not discussed in [6], but one may foresee problems when approaching $n = 3$, because the coefficients a_1 and b_1 vanish for $n = 50/17 \simeq 2.941$. In fact, $y_n^{(2,2)}$ can be seen to exactly reduce to $y_n^{(1,1)}$ at $n = 50/17$, and we may consequently expect loss of accuracy on both sides of this value.

Things are a little worse, however, because of other changes of sign in the coefficients, which yield a complicated structure for the roots of the numerator of (6), $x_{1\pm}^2 = (-a_2 \pm \sqrt{a_2^2 - 4a_1a_3})/2a_3$.

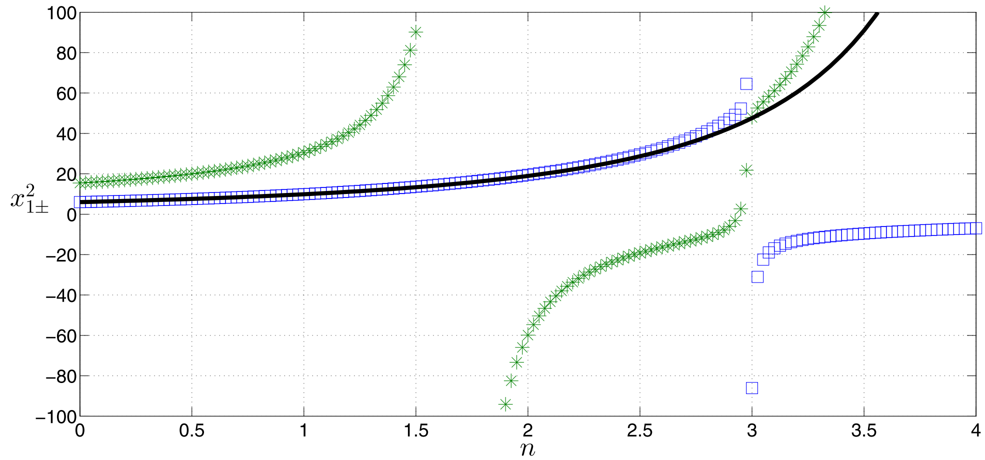

Fig. 1. The roots $x^2_{1\pm}$ of the numerator of (6) (squares for x^2_{1+}, and asterisks for x^2_{1-}), as a function of n. The black line gives reference values of x^2_1 from our Runge–Kutta numerical integrations.

In Fig. 1, these roots are displayed as functions of n (squares for x^2_{1+}, and asterisks for x^2_{1-}), and compared with values obtained through accurate Runge–Kutta integrations (black curve). It is seen that x^2_{1+} is positive only for n smaller than about 3 (a more accurate threshold being $n \simeq 3.0084$). This root yields fairly accurate values for the star radius up to $n \simeq 2.5$, as noted in [6], but then starts to diverge from the numerical values. Moreover, at $n = 50/17$, the x^2_{1-} root crosses the zero line, and for $50/17 < n < 3.0084$ both roots are positive ($0 < x^2_{1-} < x^2_{1+}$). As a consequence, the $(2, 2)$ Padé approximant completely breaks down just above $n = 50/17$. Then it recovers, and is accurate again in a very narrow interval around $n = 3$. For larger values of n, the x^2_{1-} root diverges, with an asymptote (not shown) at $n \simeq 3.705$, above which there is no positive root, and consequently no zero of y_n.

The accuracy of the $(2, 2)$ approximant for $n = 3$,

$$y_3^{(2,2)} = \frac{1 - (1/108)x^2 - (11/45\,360)x^4}{1 + (17/108)x^2 + (1/1008)x^4}\,, \tag{8}$$

which was not noted in [6], is worth being stressed, because of the practical importance of the $n = 3$ polytrope. The accuracy of (8), whose reasons are unclear, was pointed out in a later work [17], in which the $(2, 2)$ approximant for y_n was rederived, with no mention to [6] (see Table II of [17], comparing (8) with a reference numerical profile). However, neither the exceptionality of the $n = 3$ case, nor the problems of the $(2, 2)$ approximant for smaller and larger values of n were recognized in [17]. We will come back to (8) later in the paper.

3.1. Improving on the $(2, 2)$ Padé approximant

To improve on the previous results, one could construct diagonal approximants of higher order. This might get rid of the singularity near $n = 3$, and eventually extend the n domain of validity, at the expense of an increased complexity of the approximation [(3, 3) approximants for $n = 3/2$ and $n = 3$ were computed in [6], which turned out to be very accurate].

Another possibility is to look for non-diagonal approximants, even if they usually are less accurate than the diagonal ones. Using the first three terms in the Taylor expansion about the origin, we can write the $(1, 3)$ Padé approximant as

$$y_n^{(1,3)} = \frac{1 - x^2/x_1^2}{1 + b_1x^2 + b_2x^4 + b_3x^6}\,, \tag{9}$$

with

$$b_1 = \frac{1}{6} - \frac{1}{x_1^2}\,, \quad b_2 = \frac{1}{6}\left(b_1 - \frac{n}{20}\right), \tag{10}$$

$$b_3 = \frac{8n^2 - 47n + 70}{15\,120} - \frac{10 - 3n}{360}\,\frac{1}{x_1^2}\,. \tag{11}$$

We can then determine the star radius by imposing the fourth term in the Taylor expansion, as done for the $(2, 2)$ approximant, or use numerical values for it. In the former case, we find:

$$x_1^2 = \frac{6\,(12\,600 - 8460n + 1440n^2)}{12\,600 - 13\,490n + 4929n^2 - 610n^3}\,. \tag{12}$$

The resulting approximation is quite good up to $n = 2$ (not shown), but then loses accuracy, even if there is no complete breakdown before $n = 3$. Overall, it is inferior to the $(2, 2)$ Padé, but it is very accurate for some values of n. In particular, for $n = 2$, it takes the form

$$y_2^{(1,3)} = \frac{1 - \frac{19}{360}x^2}{1 + \frac{41}{360}x^2 + \frac{1}{432}x^4 - \frac{13}{226\,800}x^6}\,, \tag{13}$$

which yields a radius $x_1 = (360/19)^{1/2} = 4.352857$, extremely close to the numerical value $x_1 \simeq 4.352875$. Approximation (13) is very accurate over the whole radial extent: see Table 1, where it is compared with a reference profile computed by Horedt [4].

Table 1
Approximation (13) ($n = 2$).

x	$y(x)$ [Eq. (13)]	$y(x)$ [Horedt]
0.1	0.9983350	0.9983350
0.5	0.9593527	0.9593527
1.	0.8486541	0.8486541
3.	0.2418559	0.2418241
4.	0.0489163	0.0488401
4.3	0.0068288	0.0068109
4.35	0.0003650	0.0003660

The $(1, 3)$ Padé approximant globally improves if we release the constraint deriving from the fourth term in the Taylor expansion and place the numerical values of x_1 in (9)–(11). This causes a slight loss of accuracy at small x that is, however, amply compensated by an accuracy gain in the outer region of the star. The root mean square (RMS hereafter) difference (over $[0, x_1]$) between the resulting approximant and the numerical solution is plotted in Fig. 2 as a function of n (blue curve): it is smaller than 10^{-3} for $0 < n < 2.2$, and remains smaller than 5×10^{-3} up to $n = 3$ and slightly beyond. Errors are even smaller around some particular values of n ($n = 1.2, 2, 3$). In the same figure, we also show

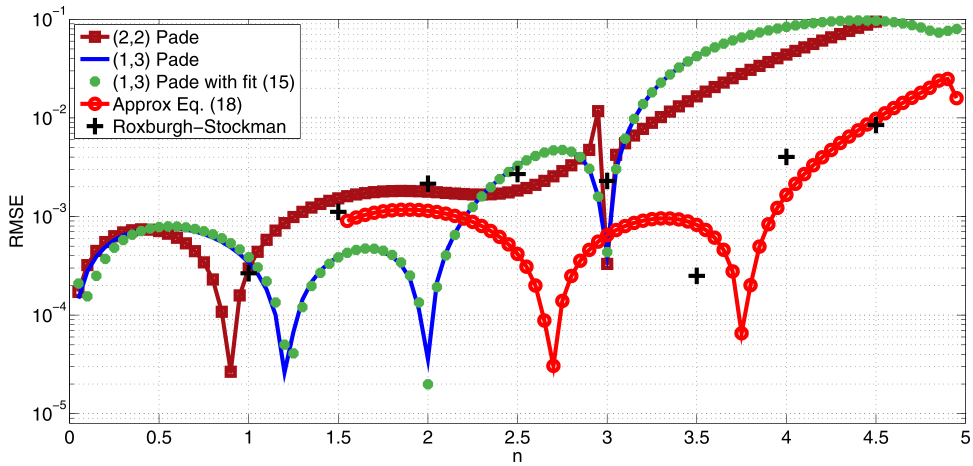

Fig. 2. Root mean square errors over $[0, x_1]$, for several approximations, computed using the numerical profiles as a reference. The brown curve refers to the $(2,2)$ Padé of [6], while the blue one is for the non-diagonal Padé approximation (9)–(11), with numerical values for x_1 [the green dots are for the same approximation, but with the fit (15) for the star radius]. The red curve is for approximation (18), and the black crosses for the best approximation given in [10]. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

Table 2
Fit (15) for the star radius.

n	$x_1(n)$ [Eq. (15)]	$x_1(n)$ [Horedt]	Error
0.5	2.7527827	2.7526980	3.1×10^{-5}
1.	3.1421898	3.1415927	1.9×10^{-4}
1.5	3.6538835	3.6537537	3.6×10^{-5}
2.	4.3521313	4.3528746	1.7×10^{-4}
2.5	5.3539359	5.3552755	2.5×10^{-4}
3.	6.8957511	6.8968486	1.6×10^{-4}
3.5	9.5343124	9.5358053	1.6×10^{-4}
4.	14.951840	14.971546	1.3×10^{-3}
4.5	31.599799	31.836463	7.4×10^{-3}

the errors for the $(2,2)$ Padé approximant (brown squares), which are significantly larger than those of the $(1,3)$ Padé approximant in the range $[1.1, 2.1]$, and then a little smaller before the breakdown, close to $n = 3$. There is also a small region around $n = 0.9$ in which the $(2,2)$ approximant is more accurate. Overall, we can say that the two approximants complement each other, and that (9)–(11), with the star radius enforced, yield a simple and fairly accurate approximation over the whole range $0 \le n \le 3.1$, which can be used as an alternative to Pascual's $(2,2)$ approximant where the latter breaks down, or loses accuracy.

A difficulty with this approach is that accurate values of x_1 in the range $[0, 4.5]$ have only been tabulated for integer and half integer n's, but this can be obviated by using a good fit for $x_1(n)$. A simple fit by Buchdahl [18],

$$x_1(n) \simeq \frac{12.3\,(1 - 0.128n)}{(5 - n)(1 - 0.15n)}, \tag{14}$$

which is within 0.5% from the tabulated reference values over $[0, 4]$, is adequate for $n > 2$, but leads to larger errors for smaller values of n (not shown). A similar result is obtained using an empirical fit by Pascual [6], whose accuracy is close to that of (14). We have therefore sought a more accurate fit by adding a quadratic term in the numerator of (14), while keeping fixed the $5 - n$ term in the denominator, which arises from analytic considerations. The new fit,

$$x_1(n) \simeq \frac{12.2378 - 1.2249n + 0.0187n^2}{(5 - n)(1 - 0.1223n)}, \tag{15}$$

has been computed from the values of x_1 resulting from our RK integrations, using a function of the MATLAB Optimization Toolbox that implements a nonlinear least-squares trust-region approach.

Table 2 shows values of x_1 obtained from (15), for integer and half-integer values of n over $[0, 4.5]$, and compares them with reference values by Horedt [3]. The fit (15) is definitely more accurate than (14), particularly for $n < 2$. Using it in (9)–(11) yields the green dots in Fig. 2, which nicely follow the curve obtained using the numerical values of the star radius.

3.2. Extending the n range of the approximation

Here, we seek analytic approximations that can be used above $n = 3$, and possibly up to $n = 5$. As we have seen, the approximation (9)–(11) does not qualify, because it rapidly becomes inaccurate above $n = 3$. A simple approximation that remains accurate beyond $n = 3$ was given in [10]. It has the form

$$y = \frac{1 - cx^2}{(1 + ex^2)^m}. \tag{16}$$

The coefficients (c, e, m) were obtained numerically, either by identifying the expansion of (16) about the origin with (5), up to the x^6 terms, or by enforcing the first term in the expansion, together with the numerical values of x_1 and $y'(x_1)$. It was shown in [10] that the latter approach gives higher accuracy. We have computed the corresponding RMS errors, using our numerical solutions as a reference, and displayed them as black crosses in Fig. 2: errors are smaller than 1% over $1 \le n \le 4.5$, confirming that this simple approximation is adequate over a wide n range.

To improve upon (16), we would like to construct an approximation of comparable accuracy and complexity that is fully analytic and does not need *a priori* knowledge of $y'(x_1)$. To do so, we need to better understand why (16) works so well. A hint comes from [11], where the singularities of the LEE in the complex plane were studied in detail. It was shown in that work that, besides a mild singularity at the star boundary, the LEE, for any n larger than unity, only has a singularity on the negative x^2 axis, at some point $x^2 = -x_s^2(n)$, with x_s a real number that determines the radius of convergence of the series expansions about the origin. Accurate values of $x_s^2(n)$ were computed numerically (see Table 1 of [11]), and are reported here in the second column of Table 3. It was also noted in [11] that the dominant singular behavior is

$$y \sim \frac{K}{[1 + (x/x_s)^2]^\sigma}, \qquad \sigma = \frac{2}{n - 1}, \tag{17}$$

with K a coefficient depending on n and x_s. This suggests that (16) might work well because it captures the dominant behavior

Table 3
Comparison between the values of $1/e$ from Eqs. (20) and (21) and the values of x_s^2 computed numerically in [11].

n	$1/e$	x_s^2
1.75	21.3083	23.0939
2	15.2001	15.7179
2.25	11.6463	11.7947
2.5	9.3597	9.3915
2.75	7.7827	7.7802
3	6.6388	6.6298
3.25	5.7765	5.7696
3.5	5.1066	5.1034
3.75	4.5731	4.5729
4	4.1393	4.1408
4.25	3.7802	3.7824
4.5	3.4785	3.4804
4.75	3.2215	3.2226
5	3.	3.
5.25	2.8070	2.8059
5.5	2.6374	2.6352
5.75	2.4872	2.4840
6	2.3532	2.3491

of y about the complex pole. Analysis of Tables 2 and 3 of [10] confirms this hypothesis; in particular, the values of $1/e$ computed from Table 2 are quite close to those of x_s^2, and the values of m are close to σ.

This leads us to consider a modification of (16) of the form

$$y = \frac{(1 - x^2/x_1^2)(1 + bx^2)}{(1 + ex^2)^\sigma}. \tag{18}$$

The radius in the numerator is enforced, as in [10], but we fix the exponent in the denominator to σ, and add another quadratic factor in the numerator, to better capture the small-x behavior. The coefficients (b, e) are determined by enforcing the first two terms of the series expansion about the origin, yielding

$$b = \sigma e - \delta, \tag{19}$$

and

$$e = \delta \frac{(n-1)}{(n-3)} \left[-1 + \sqrt{1 - \frac{n-3}{\delta^2}\left(\frac{\delta}{x_1^2} - \frac{n}{120}\right)} \right], \qquad n \neq 3, \tag{20}$$

$$e = \frac{1}{\delta}\left(\frac{1}{40} - \frac{\delta}{x_1^2}\right) \simeq 0.15063, \qquad n = 3, \tag{21}$$

with δ the positive quantity

$$\delta \equiv \frac{1}{6} - \frac{1}{x_1^2}. \tag{22}$$

Note that, for $n = 5$, letting $x_1 \to \infty$ gives $e = 1/3$ and $b = 0$, so that we recover the known analytic solution. The RMS error of approximation (18) is shown as a red curve in Fig. 2; we restrict to values above $n = 1.5$, because in the range $1 < n \leq 1.5$ there are a few points in which the value of e computed from (20) has a small imaginary component. The new approximation covers the range $1.5 < n \leq 4.5$ with good accuracy, with errors that are of about 10^{-3} or smaller up to $n = 3.9$. The red curve crosses the green one [non-diagonal Padé with fit (15) for the star radius] at about $n = 2.2$. Below this value, the approximation (9)–(11) is more accurate, whereas for larger values of n, (18) is definitely better. The new approximation is also consistently more accurate than (16), in a RMS sense, except in a neighborhood of $n = 3.5$. Thus, we have a simple, fully analytic approximation that is accurate over a wide n range, and only needs the enforcement of the value of the star radius to preserve a fair accuracy in the exterior region of the polytrope.

Because of the way (18) was constructed, we would expect the values of $1/e$ obtained from (20) and (21) to be close to the values of x_s^2 computed numerically in [11]. Table 3 shows that these values are indeed quite close, particularly for $n \geq 3$, when the singularity moves closer and closer to the origin [note that for $n \geq 5$, the values of $1/e$ are obtained by letting $x_1 \to \infty$ in (20)]. Thus, as a side-product of our approach we have got a good analytic approximation for the value of x_s over the range $1.5 < n \leq 6$.

We finally note that this approach helps us to understand why the rational approximations (8) ($n = 3$) and (13) ($n = 2$) are so good. Since $\sigma = 1$ in the former case, the purely rational approximation (8) may be expected to be accurate if it has a pole in the denominator at $x^2 \simeq -x_s^2$. This is indeed the case, because

$$1 + (17/108)x^2 + (1/1008)x^4 \simeq (1 + 0.15083x^2)(1 + 0.00658x^2), \tag{23}$$

and the first term on the rhs has a pole at $-x^2 \simeq 6.62998$, very close to the numerical value $x_s^2 = 6.6298$ computed in [11]. In the case of (13), we have $\sigma = 2$; the denominator does not vanish, but becomes very small at $x^2 \simeq -15.58$. In fact, it is quite close to

$$[1 + (x/15.58)^2]^2(1 - 0.0139x^2), \tag{24}$$

which captures well the dominant singular behavior.

4. Conclusions

After recalling some basic results, sometimes forgotten in recent literature, in this work we have focused on the construction of rational approximations to the solution of the LEE.

We have clarified merits and limitations of the $(2, 2)$ Padé approximant obtained in [6], and showed how to improve upon it. We have derived the non-diagonal Padé approximant (9)–(11), which, together with the fit (15) for the star radius, yields an accurate approximation for y over the whole radial extent of the star, for values of the polytropic index in the range $[0, 3.1]$. Building on the analysis of [11], we have also derived a modified rational approximation, meant to capture the dominant singular behavior near the complex pole of the LEE on the negative x^2 axis. The resulting approximation (18) has a simple structure, is fully analytic, and is accurate over a wide range, from about $n = 1.5$ up to $n = 4.5$. Together, the two approximations cover most of the n range of interest for star and cluster dynamics. For $n > 4.5$, (18) loses accuracy in the external region of the star, but the approximation could be improved by adding a factor $(1 + dx^2)$ in the denominator, and enforcing one more term in the series expansion about the origin.

For n close to 5, one may also use the analytic approximation derived in [12]. This approximation was obtained with a perturbation method (remember that the $n = 5$ case is exactly solvable) that is equivalent to the approach used in [8], where the expansion was performed around the exact solution of the $n = 1$ case. Beyond $n = 5$, one can only rely on the approximation derived in [7]. Further work will be needed to obtain a simple approximation in this range.

Finally, we wish to stress that all the main results on the LEE, such as those concerning the dependence of x_1 and x_s on the polytropic index, have been obtained numerically. Gaining analytic insight about these issues would be of great importance, and would likely have broader implications.

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